

Involutive Yang-Baxter Groups*

Ferran Cedó, Eric Jespers and Ángel del Río

Abstract

In 1992 Drinfeld posed the question of finding the set theoretic solutions of the Yang-Baxter equation. Recently, Gateva-Ivanova and Van den Bergh and Etingof, Schedler and Soloviev have shown a group theoretical interpretation of involutive non-degenerate solutions. Namely, there is a one-to-one correspondence between involutive non-degenerate solutions on finite sets and groups of I -type. A group \mathcal{G} of I -type is a group isomorphic to a subgroup of $\text{Fa}_n \rtimes \text{Sym}_n$ so that the projection onto the first component is a bijective map, where Fa_n is the free abelian group of rank n and Sym_n is the symmetric group of degree n . The projection of \mathcal{G} onto the second component Sym_n we call an involutive Yang-Baxter group (IYB group). This suggests the following strategy to attack Drinfeld's problem for involutive non-degenerate set theoretic solutions. First classify the IYB groups and second, for a given IYB group G , classify the groups of I -type with G as associated IYB group. It is known that every IYB group is solvable. In this paper some results supporting the converse of this property are obtained. More precisely, we show that some classes of groups are IYB groups. We also give a non-obvious method to construct infinitely many groups of I -type (and hence infinitely many involutive non-degenerate set theoretic solutions of the Yang-Baxter equation) with a prescribed associated IYB group.

1 Introduction

In a paper on statistical mechanics by Yang [14], the quantum Yang-Baxter equation appeared. It turned out to be one of the basic equations in mathematical physics and it lies at the foundation of the theory of quantum groups. One of the important unsolved problems is to discover all the solutions R of the quantum Yang-Baxter equation (note that $R : V \otimes V \rightarrow V \otimes V$, with V a vector space). In recent years, many solutions have been found and the related algebraic structures have been intensively studied (see for example [10]). Drinfeld, in [3], posed the question of finding the simplest solutions, that is, the solutions R that are induced by a bijective mapping $\mathcal{R} : X \times X \rightarrow X \times X$, where X is a basis for V . Let $\tau : X^2 \rightarrow X^2$ be the map defined by $\tau(x, y) = (y, x)$. In [5] it is shown that \mathcal{R} yields a solution if and only if the bijective mapping $r = \tau \circ \mathcal{R}$ satisfies $r_1 r_2 r_1 = r_2 r_1 r_2$, where $r_1 = r \times \text{id}_X : X^3 \rightarrow X^3$ and $r_2 = \text{id}_X \times r : X^3 \rightarrow X^3$. In this case, one says that r is a set theoretic solution of the Yang-Baxter equation.

Set theoretic solutions $r : X^2 \rightarrow X^2$ of the Yang-Baxter equation (with X a finite set) that are involutive (i.e., r^2 is the identity map on X^2) and that are (left) non-degenerate recently have

*The authors have been partially supported by grants of DGI MEC-FEDER (Spain) MTM2005-00934, Generalitat de Catalunya 2005SGR00206; Onderzoeksraad of Vrije Universiteit Brussel, Fonds voor Wetenschappelijk Onderzoek (Belgium), Flemish-Polish bilateral agreement BIL2005/VUB/06; DGI MEC-FEDER (Spain) MTM2006-06865 and Fundación Séneca of Murcia 04555/GERM/06.

2000 *Mathematics Subject Classification*. Primary 81R50, 20F29, 20B35, 20F16.

Keywords: Yang-Baxter equation, involutive non-degenerate solutions, group of I -type, finite solvable group.

received a lot of attention by Etingof, Schedler and Soloviev [5], Gateva-Ivanova and Van den Bergh [6, 7], Rump [13], Jespers and Okninski [8, 9] and others. Recall that a bijective map

$$\begin{aligned} r: X \times X &\longrightarrow X \times X \\ (x, y) &\mapsto (f_x(y), g_y(x)) \end{aligned}$$

is said to be right (respectively, left) non-degenerate if each map f_x (respectively, g_x) is bijective. Note that, since X is finite, one can show that an involutive set theoretic solution r of the Yang-Baxter equation is right non-degenerate if and only if it is left non-degenerate (see [7, Theorem 1.3], [8, Corollary 2.3] and [9, Corollary 8.2.4]).

Gateva-Ivanova and Van den Bergh in [7], and Etingof, Schedler and Soloviev in [5], gave a beautiful group theoretical interpretation of involutive non-degenerate set theoretic solutions of the Yang-Baxter equation. In order to state this, we need to introduce some notation. Let FaM_n be the free abelian monoid of rank n with basis u_1, \dots, u_n . A monoid S generated by a set $X = \{x_1, \dots, x_n\}$ is said to be of left I -type if there exists a bijection (called a left I -structure) $v: \text{FaM}_n \longrightarrow S$ such that $v(1) = 1$ and $\{v(u_1 a), \dots, v(u_n a)\} = \{x_1 v(a), \dots, x_n v(a)\}$, for all $a \in \text{FaM}_n$. In [7] it is shown that these monoids S have a presentation

$$S = \langle x_1, \dots, x_n \mid x_i x_j = x_k x_l \rangle,$$

with $\binom{n}{2}$ defining relations so that every word $x_i x_j$ with $1 \leq i, j \leq n$ appears at most once in one of the relations. Such presentation induces a bijective map $r: X \times X \longrightarrow X \times X$ defined by

$$r(x_i, x_j) = \begin{cases} (x_k, x_l), & \text{if } x_i x_j = x_k x_l \text{ is a defining relation for } S; \\ (x_i, x_j), & \text{otherwise.} \end{cases}$$

Furthermore, r is an involutive left non-degenerate set theoretical solution of the Yang-Baxter equation. Conversely, for every involutive left non-degenerate set theoretical solution of the Yang-Baxter equation $r: X \times X \longrightarrow X \times X$ and every bijection $v: \{u_1, \dots, u_n\} \rightarrow X$ there is a unique left I -structure $v: \text{FaM}_n \rightarrow S$ extending v , where S is the semigroup given by the following presentation $S = \langle X \mid ab = cd, \text{ if } r(a, b) = (c, d) \rangle$ ([9, Theorem 8.1.4.]). Furthermore, it is proved in [7] that a monoid of left I -type has a group of fractions, which is called a group of left I -type.

In [8] Jespers and Okniński proved that a monoid S is of left I -type if and only if it is of right I -type and obtained an alternative description of monoids and groups of I -type. Namely, it is shown that a monoid is of I -type if and only if it is isomorphic to a submonoid S of the semidirect product $\text{FaM}_n \rtimes \text{Sym}_n$, with the natural action of Sym_n on FaM_n (that is, $\sigma(u_i) = u_{\sigma(i)}$), so that the projection onto the first component is a bijective map, that is

$$S = \{(a, \phi(a)) \mid a \in \text{FaM}_n\},$$

for some map $\phi: \text{FaM}_n \rightarrow \text{Sym}_n$. In that case the map ϕ extends uniquely to a map $\phi: \text{Fa}_n \rightarrow \text{Sym}_n$, where Fa_n is the free abelian group of rank n , and the corresponding group of I -type SS^{-1} is isomorphic to a subgroup \mathcal{G} of the semidirect product $\text{Fa}_n \rtimes \text{Sym}_n$ so that the projection onto the first component is a bijective map, that is

$$\mathcal{G} = \{(a, \phi(a)) \mid a \in \text{Fa}_n\}. \quad (1)$$

Note that if we put $f_{u_i} = \phi(u_i)$ then $S = \langle (u_i, f_{u_i}) \mid 1 \leq i \leq n \rangle$ and one can easily obtain the associated involutive non-degenerate set theoretical solution $r: X^2 \rightarrow X^2$ defining the monoid of I -type. Indeed, if we set $X = \{u_1, \dots, u_n\}$, then $r(u_i, u_j) = (f_{u_i}(u_j), f_{f_{u_i}(u_j)}^{-1}(u_j))$. Obviously,

$\phi(\text{Fa}_n) = \langle \phi(a) \mid a \in \text{FaM}_n \rangle = \langle f_{u_i} \mid 1 \leq i \leq n \rangle$. Note that, because of Proposition 2.2 in [5], $T^{-1}f_x^{-1}T = g_x$, where $T : X \rightarrow X$ is the bijective map defined by $T(y) = f_y^{-1}(y)$. Hence $\langle f_x : x \in X \rangle$ is isomorphic with $\langle g_x : x \in X \rangle$.

So, in order to describe all involutive non-degenerate set theoretical solutions of the Yang-Baxter equation one needs to characterize the groups of I -type. An important first step in this direction is to classify the finite groups that are of the type $\phi(\text{Fa}_n)$ for some group of I -type \mathcal{G} , as in (1). A finite group with this property we will call an *involutive Yang-Baxter* (IYB, for short) group. A second step is to describe all groups of I -type that have a fixed associated IYB group G .

In [5], Etingof, Schedler and Soloviev proved that any group of I -type is solvable. As a consequence, every IYB group is solvable. In [5] it is also proved that a group \mathcal{G} is of I -type if and only if there is a bijective 1-cocycle $\mathcal{G} \rightarrow \text{Fa}_n$ with respect to some action of \mathcal{G} on Fa_n which factors through the natural action of Sym_n on $\{u_1, \dots, u_n\}$.

Now, if $\mathcal{G} = \{(a, \phi(a)) \mid a \in \text{Fa}_n\}$ is a group of I -type then the IYB group $G = \phi(\text{Fa}_n)$ naturally acts on the quotient group $A = \text{Fa}_n/K$, where $K = \{a \in \text{Fa}_n \mid \phi(a) = 1\}$ and we obtain a bijective associated 1-cocycle $G \rightarrow A$ with respect to this action. By a result of Etingof and Gelaki [4], this bijective 1-cocycle yields a non-degenerate 2-cocycle on the semidirect product $H = A \rtimes G$. This has been generalized by Ben David and Ginosar [1] to more general extensions H of A by G with a bijective 1-cocycle from G to A . This construction of Etingof and Gelaki and of Ben David and Ginosar gives rise to a group of central type in the sense of [1], i.e. a finite group H with a 2-cocycle $c \in Z^2(H, \mathbb{C}^*)$ such that the twisted group algebra $\mathbb{C}^c H$ is isomorphic to a full matrix algebra over the complex numbers, or equivalently $H = K/Z(K)$ for a finite group K with an irreducible character of degree $\sqrt{[K : Z(K)]}$. This provides a nice connection between IYB groups and groups of central type that should be investigated. The authors thank Eli Aljadeff for pointing out this connection.

It is worth mentioning that the semigroup algebra FS of a monoid of I -type S over an arbitrary field F shares many properties with the polynomial algebra in finitely many commuting variables. For example, in [7], it is shown that FS is a domain that satisfies a polynomial identity and that it is a maximal order in its classical ring of quotients. In particular, the group of I -type SS^{-1} is finitely generated abelian-by-finite and torsion free (i.e., it is a Bieberbach group). The homological properties for FS were the main reasons for studying monoids of I -type in [7] and it was inspired by earlier work of Tate and Van den Bergh on Sklyanin algebras.

In this paper we investigate group theoretical properties of IYB groups. The content of the paper is as follows. In Section 2 we obtain several characterizations of IYB groups. These allow us in Section 3 to prove that the class of IYB groups includes the following: finite abelian-by-cyclic groups, finite nilpotent groups of class 2, direct products and wreath products of IYB groups, semidirect products $A \rtimes H$ with A a finite abelian group A and H an IYB group, Hall subgroups of IYB groups, Sylow subgroups of symmetric groups Sym_n . These results imply that any finite nilpotent group is a subgroup of an IYB (nilpotent) group. It is unclear whether the class of IYB groups is closed for taking subgroups. As a consequence, we do not know whether the class of IYB groups contains all finite nilpotent groups. At this point it nevertheless is tempting to conjecture that the class of IYB groups coincides with that of all solvable finite groups. To prove this, one would like to be able to lift the IYB structure from subgroups H or quotient groups \overline{G} of a given group G to G . In Section 4, we give some examples of IYB groups of minimal order that are not covered by the general results of Section 3. The last example, a 3-group of class 3, shows that not every IYB homomorphism (see Section 2 for the definition) of a quotient of G can be lifted to an IYB homomorphism of G . This indicates that there is no obvious inductive process to prove that nilpotent finite groups are IYB. In Section 5 we consider the connection between set

theoretic solutions of the Yang-Baxter equation and IYB groups from a different perspective. If $r(x_1, x_2) = (f_{x_1}(x_2), g_{x_2}(x_1))$ is an involutive non-degenerate solution on a finite set X of the set-theoretical Yang-Baxter equation then it is easy to produce, in an obvious manner, infinitely many solutions with the same associated IYB group, namely for every set Y let $r_Y : (X \cup Y)^2 \rightarrow (X \cup Y)^2$ be given by $r_Y((x_1, y_1), (x_2, y_2)) = ((f_{x_1}(x_2), y_1), (g_{x_2}(x_1), y_2))$. We show an alternative way of obtaining another IYB map on $X \times X$ with the same associated IYB group. Hence providing, in a non-obvious fashion, infinitely many set theoretic solutions for the same IYB group.

2 A characterization of IYB groups

In this section we obtain several characterizations of IYB groups. In order to state these we introduce the following terminology. For a finite set X we denote by Sym_X the symmetric group on X . An *involutive Yang-Baxter map* (IYB map, for short) on a finite set X is a map $\lambda : X \rightarrow \text{Sym}_X$ satisfying

$$\lambda(x)\lambda(\lambda(x)^{-1}(y)) = \lambda(y)\lambda(\lambda(y)^{-1}(x)) \quad (x, y \in X). \quad (2)$$

The justification for this terminology is based on the fact that each IYB map yields an involutive non degenerate set theoretical solution of the Yang-Baxter equation and conversely. Indeed, let $r : X^2 \rightarrow X^2$ be a bijective map. As before, denote $r(x, y) = (f_x(y), g_y(x))$. From the proof of [2, Theorem 4.1], it follows that $r : X^2 \rightarrow X^2$ is an involutive non degenerate set theoretical solution of the Yang-Baxter equation if and only if $f_x \in \text{Sym}_X$ for all $x \in X$ and the map $\lambda : X \rightarrow \text{Sym}_X$ defined by $\lambda(x) = f_x$, for all $x \in X$, is an IYB map.

Theorem 2.1 *The following conditions are equivalent for a finite group G .*

1. G is an IYB group, that is, there is a map $\phi : \text{Fa}_n \rightarrow \text{Sym}_n$ such that $\{(a, \phi(a)) : a \in \text{Fa}_n\}$ is a subgroup of $\text{Fa}_n \rtimes \text{Sym}_n$ and G is isomorphic to $\phi(\text{Fa}_n)$.
2. There is an abelian group A , an action of G on A and a group homomorphism $\rho : G \rightarrow A \rtimes G$ such that $\pi_G \rho = \text{id}_G$ and $\pi_A \rho : G \rightarrow A$ is bijective, where π_G and π_A are the natural projections on G and A respectively.
3. There is an abelian group A , an action of G on A and a bijective 1-cocycle $G \rightarrow A$.
4. There exists an IYB map $\lambda : A \cup X \rightarrow \text{Sym}_{A \cup X}$ satisfying the following conditions:
 - (a) $\lambda(A)$ is a subgroup of $\text{Sym}_{A \cup X}$ isomorphic to G ,
 - (b) $A \cap X = \emptyset$,
 - (c) $\lambda(x) = \text{id}_{A \cup X}$ for all $x \in X$,
 - (d) $\lambda(a)(b) \in A$ for all $a, b \in A$ and
 - (e) $\lambda|_A$ is injective.
5. $G \cong \lambda(X)$ for some IYB map $\lambda : X \rightarrow \text{Sym}_X$ whose image is a subgroup of Sym_X .
6. $G \cong \langle \lambda(X) \rangle$ for some IYB map $\lambda : X \rightarrow \text{Sym}_X$.
7. There exist a group homomorphism $\mu : G \rightarrow \text{Sym}_G$ satisfying

$$x\mu(x)^{-1}(y) = y\mu(y)^{-1}(x), \quad (3)$$

for all $x, y \in G$.

8. There exist a generating subset Z of G and a group homomorphism $\mu : G \rightarrow \text{Sym}_Z$ satisfying (3) for all $x, y \in Z$.

Proof. 1 implies 2. Suppose that G is an IYB group. Thus there is a map $\phi : \text{Fa}_n \rightarrow \text{Sym}_n$ such that $\mathcal{G} = \{(a, \phi(a)) \mid a \in \text{Fa}_n\}$ is a subgroup of $\text{Fa}_n \rtimes \text{Sym}_n$, and $G \cong \phi(\text{Fa}_n)$. We may assume that $G = \phi(\text{Fa}_n)$. Let $K = \{(a, \phi(a)) \in \mathcal{G} \mid \phi(a) = 1\}$. Then G is isomorphic to \mathcal{G}/K and $B = \{b \in \text{Fa}_n \mid (b, 1) \in K\}$ is a subgroup of Fa_n . Furthermore, if $b \in B$ and $a \in \text{Fa}_n$ then

$$(a, \phi(a))(b, 1)(a, \phi(a))^{-1} = (a\phi(a)(b), \phi(a))(\phi(a)^{-1}(a^{-1}), \phi(a)^{-1}) = (\phi(a)(b), 1) \in K.$$

This shows that B is invariant by the restriction to G of the action of Sym_n on Fa_n . Hence this action induces an action of G on $A = \text{Fa}_n/B$. Furthermore, $(b, 1)(a, \phi(a)) = (ba, \phi(a))$ and so $\phi(ba) = \phi(a)$ for each $b \in B$ and $a \in \text{Fa}_n$. Let $a, c \in \text{Fa}_n$ such that $\phi(a) = \phi(c)$. Then

$$\phi(a) = \phi(cc^{-1}a) = \phi(c)\phi(\phi(c)^{-1}(c^{-1}a)).$$

Hence $\phi(\phi(c)^{-1}(c^{-1}a)) = 1$, that is $\phi(c)^{-1}(c^{-1}a) \in B$. Since B is invariant under the action of the elements of G , we have $c^{-1}a \in B$, i.e. $\phi(c^{-1}a) = 1$. Thus ϕ induces a bijective map $\lambda : A \rightarrow G$. Furthermore the map $\rho : G \rightarrow A \rtimes G$ defined by $\rho(g) = (\lambda^{-1}(g), g)$ satisfies the required conditions.

2 implies 3. Let G, A and $\rho : G \rightarrow A \rtimes G$ as in 2. Thus $\rho(G) = \{(\pi(g), g) : g \in G\}$ is a subgroup of $A \rtimes G$, with $\pi = \pi_A \rho$. Then $(\pi(gh), gh) = (\pi(g), g)(\pi(h), h) = (\pi(g)g(\pi(h)), gh)$ and hence π is a 1-cocycle and it is bijective by assumption.

3 implies 4. Let G act on an abelian group A and let $\pi : G \rightarrow A$ be a bijective 1-cocycle. If $g, h \in G$ then

$$g\pi^{-1}(g^{-1}\pi(h)) = \pi^{-1}\pi(g\pi^{-1}(g^{-1}\pi(h))) = \pi^{-1}(\pi(g)g\pi^{-1}g^{-1}\pi(h)) = \pi^{-1}(\pi(g)\pi(h)).$$

Therefore

$$g\mu(g)^{-1}(h) = h\mu(h)^{-1}(g), \quad (g, h \in G), \quad (4)$$

because A is abelian, where $\mu(g) : G \rightarrow G$ is the map given by $\mu(g)(h) = \pi^{-1}(g\pi(h))$, for $g, h \in G$.

Let $\psi : G \rightarrow \text{Sym}_X$ be a monomorphism, where X is a finite set such that $A \cap X = \emptyset$. Let $\nu : A \cup X \rightarrow G$ be the map defined by

$$\nu(t) = \begin{cases} \pi^{-1}(t), & \text{if } t \in A; \\ 1, & \text{if } t \in X. \end{cases}$$

Let $\varphi : G \rightarrow \text{Sym}_{A \cup X}$ be the map such that $\varphi(g)$ acts as g on A and as $\psi(g)$ on X . It is clear that φ is a monomorphism. Clearly $\lambda = \varphi\nu$ satisfies conditions (a)-(e) and we shall see that λ is an IYB map, that is, it satisfies condition (2). If $x \in X$ then $\lambda(x) = \lambda(\lambda(y)^{-1}(x)) = 1$ and thus the equality in (2) holds for every $y \in A \cup X$. Similarly the equality holds if $y \in X$. Finally, assume that $x, y \in A$. In this case $\pi^{-1}(x) \pi^{-1}(\pi^{-1}(x)^{-1}(y)) = \pi^{-1}(y) \pi^{-1}(\pi^{-1}(y)^{-1}(x))$, by (4), and the equality (2) follows by the following calculation

$$(\lambda(x) \lambda(\lambda(x)^{-1}(y)))(v) = \begin{cases} \pi^{-1}(x) \pi^{-1}(\pi^{-1}(x)^{-1}(y))(v), & \text{if } v \in A; \\ \psi(\pi^{-1}(x) \pi^{-1}(\pi^{-1}(x)^{-1}(y)))(v), & \text{if } v \in X. \end{cases}$$

4 implies 5 is obvious.

5 implies 7. Suppose that there exists an IYB map $\lambda : X \rightarrow \text{Sym}_X$ such that $\lambda(X)$ is a subgroup of Sym_X and $G \cong \lambda(X)$. We may assume that $G = \lambda(X)$. Define $\mu : G \rightarrow \text{Sym}_G$ by

$$\mu(g)(\lambda(x)) = \lambda(g(x)),$$

for all $g \in G$ and all $x \in X$. We shall see that μ is well defined. Let $x, y \in X$ such that $\lambda(x) = \lambda(y)$ and let $g \in G$. Since $G = \lambda(X)$, there exists $z \in X$ such that $g = \lambda(z)^{-1}$. Since λ is an IYB map,

$$\lambda(z) \lambda(\lambda(z)^{-1}(x)) = \lambda(x) \lambda(\lambda(x)^{-1}(z)) = \lambda(y) \lambda(\lambda(y)^{-1}(z)) = \lambda(z) \lambda(\lambda(z)^{-1}(y)).$$

So $\lambda(\lambda(z)^{-1}(x)) = \lambda(\lambda(z)^{-1}(y))$, that is $\lambda(g(x)) = \lambda(g(y))$. Hence μ is well defined.

Let $g, h \in G$ and $x \in X$. We have

$$\mu(gh)(\lambda(x)) = \lambda(gh(x)) = \mu(g)(\lambda(h(x))) = \mu(g)(\mu(h)(\lambda(x))).$$

Therefore μ is a group homomorphism. Furthermore, there exist $y, z \in X$ such that $g = \lambda(y)$ and $h = \lambda(z)$, and we have

$$g \mu(g)^{-1}(h) = \lambda(y) \lambda(\lambda(y)^{-1}(z)) = \lambda(z) \lambda(\lambda(z)^{-1}(y)) = h \mu(h)^{-1}(g).$$

7 implies 8 is obvious.

8 implies 6. Let $\mu : G = \langle Z \rangle \rightarrow \text{Sym}_Z$ be a map satisfying condition 8. Let $\alpha : G \rightarrow \text{Sym}_Y$ be a monomorphism, for some finite set Y such that $Y \cap Z = \emptyset$. Let $X = Y \cup Z$. Let $\lambda : X \rightarrow \text{Sym}_X$ be the map defined by

$$\lambda(y)(x) = x, \quad \lambda(z)(y) = \alpha(z)(y) \quad \text{and} \quad \lambda(z)(z') = \mu(z)(z'),$$

for all $x \in X$, $y \in Y$ and $z, z' \in Z$.

Since α and μ are group homomorphisms, the restriction of λ to Z , $\lambda|_Z : Z \rightarrow \text{Sym}_X$, extends to a group homomorphism $f : G \rightarrow \text{Sym}_X$ which is injective, because so is α . Thus $G \cong f(G) = \langle \lambda(X) \rangle$, and so it is enough to show that λ is an IYB map, i.e. condition (2) holds for every $x, y \in X$. If $x, y \in Y$ then

$$\lambda(x) = \lambda(y) = \lambda(\lambda(x)^{-1}(y)) = \lambda(\lambda(y)^{-1}(x)) = \text{id}_X$$

and condition (2) follows easily in this case. If $x \in Y$ and $y \in Z$ then

$$\lambda(x) = \lambda(\lambda(y)^{-1}(x)) = \text{id}_X.$$

Hence

$$\lambda(x) \lambda(\lambda(x)^{-1}(y)) = \lambda(y) = \lambda(y) \lambda(\lambda(y)^{-1}(x)).$$

Finally, if $x, y \in Z$ then $\lambda(x)^{-1}(y) = \mu(x)^{-1}(y)$ and $\lambda(y)^{-1}(x) = \mu(y)^{-1}(x)$. Thus $x \lambda(x)^{-1}(y) = x \mu(x)^{-1}(y) = y \mu(y)^{-1}(x) = y \lambda(y)^{-1}(x)$. Then

$$\lambda(x) \lambda(\lambda(x)^{-1}(y)) = f(x \lambda(x)^{-1}(y)) = f(y \lambda(y)^{-1}(x)) = \lambda(y) \lambda(\lambda(y)^{-1}(x)),$$

as wanted.

6 implies 1. Let $\lambda : X \rightarrow \text{Sym}_X$ be an IYB map such that $G \cong \langle \lambda(X) \rangle$. Let FaM_X be the free abelian monoid with basis X . We extend λ to a map $\lambda : \text{FaM}_X \rightarrow \text{Sym}_X$ by setting

$$\lambda(1) = 1 \quad \text{and} \quad \lambda(xa) = \lambda(x) \lambda(\lambda(x)^{-1}(a)), \quad \text{if } x \in X \text{ and } a \in \text{FaM}_X,$$

where the action of Sym_X on FaM_X is the natural one. We need to show that λ is well defined or equivalently that if $a = xb = yc$ with $x, y \in X$ and $b, c \in \text{FaM}_X$ then $\lambda(x) \lambda(\lambda(x)^{-1}(b)) = \lambda(y) \lambda(\lambda(y)^{-1}(c))$. We may assume that $x \neq y$. In that case there is $d \in \text{FaM}_X$ such that $b = yd$ and $c = xd$. We argue on $l(d)$, where the map $l : \text{FaM}_X \rightarrow \mathbb{N}$ is defined by $l(\prod_{x \in X} x^{m_x}) =$

$\sum_{x \in X} m_x$, ($m_x \geq 0$). Since λ is an IYB map, the claim follows if $l(d) = 0$. In the induction argument we use that $\lambda(a_1)$ is well defined if $l(a_1) < l(a)$.

$$\begin{aligned}
\lambda(x) \lambda(\lambda(x)^{-1}(b)) &= \lambda(x) \lambda(\lambda(x)^{-1}(y) \lambda(x)^{-1}(d)) && \text{(because } \lambda(x) \text{ is a isomomorphism)} \\
&= \lambda(x) \lambda(\lambda(x)^{-1}(y)) \lambda(\lambda(\lambda(x)^{-1}(y))^{-1}(\lambda(x)^{-1}(d))) && \text{(by induction)} \\
&= \lambda(x) \lambda(\lambda(x)^{-1}(y)) \lambda([\lambda(x) \lambda(\lambda(x)^{-1}(y))]^{-1}(d)) \\
&= \lambda(y) \lambda(\lambda(y)^{-1}(x)) \lambda([\lambda(y) \lambda(\lambda(y)^{-1}(x))]^{-1}(d)) && \text{(by hypothesis)} \\
&= \lambda(y) \lambda(\lambda(y)^{-1}(x)) \lambda(\lambda(\lambda(y)^{-1}(x))^{-1}(\lambda(y)^{-1}(d))) \\
&= \lambda(y) \lambda(\lambda(y)^{-1}(x) \lambda(y)^{-1}(d)) && \text{(by induction)} \\
&= \lambda(y) \lambda(\lambda(y)^{-1}(c)) && \text{(because } \lambda(y) \text{ is a isomomorphism).}
\end{aligned}$$

So λ is well defined.

Now we show that $\lambda(ab) = \lambda(a)\lambda(\lambda(a)^{-1}(b))$, for every $a, b \in \text{FaM}_X$. Again we argue by induction on $l(a)$. The case $l(a) \leq 1$ follows by the definition of λ and the induction step follows by the following computation, with $x \in X$:

$$\begin{aligned}
\lambda(xa \cdot b) &= \lambda(x) \lambda(\lambda(x)^{-1}(ab)) = \lambda(x) \lambda(\lambda(x)^{-1}(a) \lambda(x)^{-1}(b)) \\
&= \lambda(x) \lambda(\lambda(x)^{-1}(a)) \lambda(\lambda(\lambda(x)^{-1}(a))^{-1}(\lambda(x)^{-1}(b))) \\
&= \lambda(x) \lambda(\lambda(x)^{-1}(a)) \lambda([\lambda(x) \lambda(\lambda(x)^{-1}(a))]^{-1}(b)) \\
&= \lambda(xa) \lambda(\lambda(xa)^{-1}(b)).
\end{aligned}$$

By [9, Lemma 8.2.2], λ extends uniquely to a map $\lambda: \text{Fa}_X \rightarrow \text{Sym}_X$ such that

$$\lambda(1) = 1 \quad \text{and} \quad \lambda(ab) = \lambda(a)\lambda(\lambda(a)^{-1}(b))$$

for every $a, b \in \text{Fa}_X$, where Fa_X is the free abelian group with basis X . Now it is easy to see that $\{(a, \lambda(a)) \mid a \in \text{Fa}_X\}$ is a subgroup of $\text{Fa}_X \rtimes \text{Sym}_X$. Thus $\lambda(\text{Fa}_X)$ is an IYB group. By the definition of λ one has $\langle \lambda(X) \rangle = \lambda(\text{Fa}_X)$ and so G is an IYB group. ■

Let G be a finite group. A group homomorphism $\mu: G \rightarrow \text{Sym}_G$ satisfying condition (3) we call an *IYB morphism* of G . So, because of Theorem 2.1, a finite group is an IYB group if and only if it admits an IYB morphism, or equivalently, if it admits a bijective 1-cocycle $G \rightarrow A$ with respect to some action of G on an abelian group A . In the remainder of the paper these two conditions turn out to be very useful. It therefore is convenient to know how to pass directly from one to the other.

If $\pi: G \rightarrow A$ is a bijective 1-cocycle then the equality in (4) holds for every $g, h \in G$. Equivalently, for every $g, h \in G$,

$$g(\pi^{-1} \circ \alpha_g^{-1} \circ \pi)(h) = h(\pi^{-1} \circ \alpha_h^{-1} \circ \pi)(g),$$

where $\alpha_g: A \rightarrow A$ is the map given by $\alpha_g(a) = g(a)$, for $a \in A$. Thus $g \mapsto \pi^{-1} \circ \alpha_g \circ \pi$ is an IYB morphism of G . (Notice that the proof of (4) does not use that α_g is a group homomorphism.

Thus even if α_g is not assumed to be a group homomorphism for all g , still this defines an IYB morphism.)

Conversely, let μ be an IYB morphism of G . We define a new product $*$ on G . So let $a, b \in G$. Put

$$a * b = a\mu(a)^{-1}(b) = b\mu(b)^{-1}(a).$$

We claim that $A = (G, *)$ is an abelian group, μ gives an action of G on A and the identity $1 : G \rightarrow A$ is a bijective 1-cocycle. That $\mu : G \rightarrow \text{Sym}_A$ is a group homomorphism and $1(gh) = 1(g)\mu(g)(1(h))$ is clear. To prove the associativity of $*$ we first prove the following equality:

$$\mu(x)(yz) = \mu(x)(y) \cdot \mu(\mu(y)^{-1}(x^{-1}))^{-1}(z) \quad (x, y, z \in G). \quad (5)$$

Indeed

$$\begin{aligned} \mu(x)(yz) &= xx^{-1}\mu(x^{-1})^{-1}(yz) = xyz\mu(yz)^{-1}(x^{-1}) = xyz\mu(z)^{-1}(\mu(y)^{-1}(x^{-1})) \\ &= xy\mu(y)^{-1}(x^{-1})\mu(\mu(y)^{-1}(x^{-1}))^{-1}(z) = xx^{-1}\mu(x^{-1})^{-1}(y)\mu(\mu(y)^{-1}(x^{-1}))^{-1}(z) . \\ &= \mu(x)(y)\mu(\mu(y)^{-1}(x^{-1}))^{-1}(z) \end{aligned}$$

Now, if $x, y, z \in A$ then

$$\begin{aligned} x * (y * z) &= x\mu(x^{-1})(y\mu(y^{-1})(z)) = x\mu(x^{-1})(y)\mu(\mu(y^{-1})(x))^{-1}(\mu(y^{-1})(z)) \quad (\text{by (5)}) \\ &= x\mu(x^{-1})(y)\mu((y\mu(y^{-1})(x))^{-1}(z)) = (x * y)\mu(x * y)^{-1}(z) = (x * y) * z. \end{aligned}$$

and the associativity of $*$ follows. The verification of the remaining axioms of group are straightforward.

In this paper (as customary), it is assumed that all actions of a group G on an abelian group A are via automorphisms on A . Hence, we should now verify that $\mu(g)$ is an automorphism of A , for each $g \in G$. However, because of the presence of the bijective 1-cocycle it turns out that this property automatically is satisfied. More precisely we prove the following result.

Proposition 2.2 *Let G and A be two groups with A abelian. Let $\alpha : G \rightarrow \text{Sym}_A$ be a group homomorphism and assume that there is a bijection $\pi : G \rightarrow A$ satisfying*

$$\pi(gh) = \pi(g)\alpha(g)(\pi(h)) \quad \text{for every } g, h \in G. \quad (6)$$

Then α defines an action (by automorphisms) of G on A .

Proof. Let α_g denote the image of $g \in G$ by α . We have to show that α_g is a group homomorphism for every $g \in G$. We define $\mu : G \rightarrow \text{Sym}_G$ by $\mu(g) = \pi^{-1}\alpha_g\pi$. As it was pointed above, even if α_g is not assumed to be a group homomorphism, μ is an IYB morphism. In particular μ satisfies (5). Then

$$\begin{aligned} \alpha_g(ab) &= \alpha_g(\pi(\pi^{-1}(a))\alpha_{\pi^{-1}(a)}\pi\pi^{-1}\alpha_{\pi^{-1}(a)}^{-1}(b)) \\ &= \alpha_g(\pi(\pi^{-1}(a)\pi^{-1}\alpha_{\pi^{-1}(a)}^{-1}(b))) & (6) \\ &= \pi\mu(g)(\pi^{-1}(a)\pi^{-1}\alpha_{\pi^{-1}(a)}^{-1}(b)) & (\text{definition of } \mu) \\ &= \pi(\mu(g)(\pi^{-1}(a))\mu(\mu(\pi^{-1}(a))^{-1}(g^{-1}))^{-1}\pi^{-1}\alpha_{\pi^{-1}(a)}^{-1}(b)) & (5) \\ &= \pi(\mu(g)(\pi^{-1}(a))\pi^{-1}\alpha_{\mu(\pi^{-1}(a))^{-1}(g^{-1})}^{-1}\alpha_{\pi^{-1}(a)}^{-1}(b)) & (\text{definition of } \mu) \\ &= \pi(\mu(g)(\pi^{-1}(a))\pi^{-1}\alpha_{\pi^{-1}(a)}^{-1}\alpha_{\mu(\pi^{-1}(a))^{-1}(g^{-1})}(b)) & (\alpha \text{ is homomorphism}) \\ &= \pi(\mu(g)(\pi^{-1}(a))\pi^{-1}\alpha_{g^{-1}\mu(g)(\pi^{-1}(a))}^{-1}(b)) & (\mu \text{ is IYB morphism}) \\ &= \pi(\mu(g)(\pi^{-1}(a)))\alpha_{\mu(g)(\pi^{-1}(a))}\pi\pi^{-1}\alpha_{g^{-1}\mu(g)(\pi^{-1}(a))}^{-1}(b) & (6) \\ &= \alpha_g(a)\alpha_g(b) & (\alpha \text{ is homomorphism}) \end{aligned}$$

■

3 The class of IYB groups

In this section we prove that some classes of groups consist of IYB groups and that the class of IYB groups is closed under some constructions. We start with some easy consequences of the characterization of IYB groups in terms of 1-cocycles.

Recall from [5, Theorem 2.15] that every group of I -type is solvable, and henceforth so is every IYB group. For the sake of completeness, we include a proof, which is essentially the proof of [5, Theorem 2.15]. Indeed, if $\pi : G \rightarrow A$ is a bijective 1-cocycle then the 1-cocycle condition implies that if B is a characteristic subgroup of A then $\pi^{-1}(B)$ is a subgroup of G . In particular, if p is a prime and P is the Hall p' -subgroup of A , then $\pi^{-1}(P)$ is a Hall p' -subgroup of G . By a theorem of P. Hall [12, 9.1.8], G is solvable.

Corollary 3.1 *If G is an IYB group then its Hall subgroups are also IYB.*

Proof. Assume that G is an IYB group. By Theorem 2.1 there is a bijective 1-cocycle $\pi : G \rightarrow A$, for some abelian group A . If B is a Hall subgroup of A then B is invariant under the action of G and hence $H = \pi^{-1}(B)$ is a subgroup of G of the same order than B . Then the restriction of π to H , $\pi|_H : H \rightarrow B$, is a bijective 1-cocycle, with respect to the action of H restricted to B . ■

Corollary 3.2 *The class of IYB groups is closed under direct products.*

Proof. Let G_1 and G_2 be IYB groups. By Theorem 2.1 there are bijective 1-cocycles $\pi_i : G_i \rightarrow A_i$ with respect to some action of G_i on an abelian group A_i ($i = 1, 2$). Then $\pi_1 \times \pi_2 : G_1 \times G_2 \rightarrow A_1 \times A_2$ is a bijective 1-cocycle with respect to the obvious action of $G_1 \times G_2$ on $A_1 \times A_2$, namely $(g_1, g_2)(a_1, a_2) = (g_1(a_1), g_2(a_2))$. ■

The next two results provide more closure properties of the class of IYB groups.

Theorem 3.3 *Let G be a finite group such that $G = AH$, where A is an abelian normal subgroup of G and H is an IYB subgroup of G . Suppose that there is a bijective 1-cocycle $\pi : H \rightarrow B$, with respect to an action of H on the abelian group B such that $H \cap A$ acts trivially on B . Then G is an IYB group.*

In particular, every semidirect product $A \rtimes H$ of a finite abelian group A by an IYB group H is IYB.

Proof. Let $N = \{(h^{-1}, \pi(h)) \in H \times B \mid h \in H \cap A\}$. Since $H \cap A$ acts trivially on B , $\pi(ah) = \pi(a)\pi(h)$ for every $a \in A \cap H$ and $h \in H$. It follows that N is a subgroup of $A \times B$.

Let $C = (A \times B)/N$ and let $\overline{(a, b)}$ denote the class of $(a, b) \in A \times B$ modulo N . Note that $|G| = |C|$. Define the following action of G on C :

$$\overline{g(a, b)} = \overline{(gag^{-1}, h(b))},$$

for all $a \in A$, $b \in B$ and $g = a'h \in G$, with $a' \in A$ and $h \in H$. We shall see that this is well defined and it indeed is an action. Let $a, a' \in A$ and $b, b' \in B$ such that $\overline{(a, b)} = \overline{(a', b')}$. We have $h^{-1} = a^{-1}a' \in H \cap A$ and $\pi(h) = b^{-1}b'$. Let $g = a_1h_1 = a_2h_2 \in G$, with $a_1, a_2 \in A$ and $h_1, h_2 \in H$. Since A is an abelian normal subgroup of G , we have

$$(gag^{-1})^{-1}ga'g^{-1} = h_1a^{-1}a'h_1^{-1} = h_1h^{-1}h_1^{-1} \in H \cap A$$

and

$$h_2^{-1}h_1 = h_2^{-1}(h_1h_2^{-1})h_2 = h_2^{-1}(a_1^{-1}a_2)h_2 \in H \cap A.$$

Since $h, h_2^{-1}h_1 \in H \cap A$ and $H \cap A$ acts trivially on B , we have

$$\begin{aligned} \pi(((gag^{-1})^{-1}ga'g^{-1})^{-1}) &= \pi(h_1hh_1^{-1}) = \pi(h_1)h_1(\pi(hh_1^{-1})) \\ &= \pi(h_1)h_1(\pi(h)\pi(h_1^{-1})) = \pi(h_1)h_1(\pi(h))h_1(\pi(h_1^{-1})) \\ &= \pi(h_1)h_1(\pi(h_1^{-1}))h_1(\pi(h)) = \pi(h_1h_1^{-1})h_1(\pi(h)) \\ &= h_1(\pi(h)) = h_1(b^{-1}b') \\ &= h_1(b)^{-1}h_1(b') = h_2(b)^{-1}h_1(b'). \end{aligned}$$

Hence $\overline{(gag^{-1}, h_2(b))} = \overline{(ga'g^{-1}, h_1(b'))}$. Furthermore, if $g, g' \in G$, $g = bh$ and $g' = b'h'$, with $b, b' \in A$ and $h, h' \in H$, then we have $gg' = (bhb'h^{-1})hh'$ and

$$(gg')\overline{(a, b)} = \overline{((gg')a(gg')^{-1}, (hh')(b))} = \overline{(g(g'a g'^{-1})g^{-1}, h(h'(b)))} = g(g'\overline{(a, b)}).$$

Therefore we have a well defined action of G on C .

We define $\bar{\pi}: G \rightarrow C$ by

$$\bar{\pi}(g) = \overline{(a, \pi(h))}$$

for all $g \in G$ with $g = ah$, where $a \in A$ and $h \in H$. Note that if $g = ah = a'h'$ with $a, a' \in A$ and $h, h' \in H$, then $a^{-1}a' = hh'^{-1} \in H \cap A$, $h^{-1}h' = h'^{-1}(hh'^{-1})^{-1}h' \in H \cap A$ and hence

$$\begin{aligned} \pi((a^{-1}a')^{-1}) &= \pi(h'h^{-1}) = \pi(h')h'(\pi(h^{-1})) \\ &= \pi(h')hh^{-1}h'(\pi(h^{-1})) = \pi(h')h(\pi(h^{-1})) \\ &= \pi(h')\pi(h)^{-1}, \end{aligned}$$

because $H \cap A$ acts trivially on B . Hence $\bar{\pi}$ is well defined. Let $g_1 = a_1h_1$ and $g_2 = a_2h_2$, with $a_1, a_2 \in A$ and $h_1, h_2 \in H$. We have

$$\begin{aligned} \bar{\pi}(g_1g_2) &= \bar{\pi}((a_1h_1a_2h_1^{-1})h_1h_2) = \overline{(a_1h_1a_2h_1^{-1}, \pi(h_1h_2))} \\ &= \overline{(a_1h_1a_2h_1^{-1}, \pi(h_1)h_1(\pi(h_2)))} = \overline{(a_1, \pi(h_1))} \overline{(h_1a_2h_1^{-1}, h_1(\pi(h_2)))} \\ &= \overline{(a_1, \pi(h_1))}g_1\overline{(a_2, \pi(h_2))} = \bar{\pi}(g_1)g_1(\bar{\pi}(g_2)). \end{aligned}$$

Hence $\bar{\pi}$ is a bijective 1-cocycle and we conclude that G is an IYB group, by Theorem 2.1. \blacksquare

Theorem 3.4 *Let N and H be IYB groups and let $\pi_N : N \rightarrow A$ be a bijective 1-cocycle with respect to an action of N on an abelian group A . If $\gamma : H \rightarrow \text{Aut}(N)$ and $\delta : H \rightarrow \text{Aut}(A)$ are actions of H on N and A respectively such that $\delta(h)\pi_N = \pi_N\gamma(h)$ for every $h \in H$, then the semidirect product $N \rtimes H$, with respect to the action γ , is an IYB group.*

Proof. Let $\alpha : N \rightarrow \text{Aut}(A)$ be an action of N on the abelian group A such that $\pi_N : N \rightarrow A$ is a bijective 1-cocycle. Let $\pi_H : H \rightarrow B$ be a bijective 1-cocycle with respect to the action $\beta : H \rightarrow \text{Aut}(B)$. We are going to denote by $\alpha_n, \beta_h, \gamma_h$ and δ_h to the images of $n \in N$ or $h \in H$ under the respective maps α, β, γ and δ .

If $h \in H$ and $n_1, n_2 \in N$ then

$$\begin{aligned} \delta_h\pi_N(n_1) \delta_h\alpha_{n_1}(\pi_N(n_2)) &= \delta_h(\pi_N(n_1) \alpha_{n_1}(\pi_N(n_2))) = \delta_h\pi_N(n_1n_2) = \pi_N\gamma_h(n_1n_2) \\ &= \pi_N(\gamma_h(n_1)\gamma_h(n_2)) = \pi_N\gamma_h(n_1)\alpha_{\gamma_h(n_1)}(\pi_N\gamma_h(n_2)) \\ &= \delta_h\pi_N(n_1)\alpha_{\gamma_h(n_1)}\delta_h(\pi_N(n_2)). \end{aligned}$$

This shows that

$$\delta_h \alpha_n = \alpha_{\gamma_h(n)} \delta_h \quad (7)$$

for every $h \in H$ and $n \in N$.

Now we define the following map $\sigma : G = N \rtimes H \rightarrow \text{Aut}(A \times B)$:

$$\sigma_{nh}(a, b) = (\alpha_n \delta_h(a), \beta_h(b)),$$

for all $n \in N$ and $h \in H$. Since both α_n and δ_h are automorphisms of A and β_h is an automorphism of B , σ_{nh} is an automorphism of $A \times B$. Now we check that σ is a group homomorphism. Let $n_1, n_2 \in N$ and $h_1, h_2 \in H$. Then

$$\begin{aligned} \sigma_{n_1 h_1 n_2 h_2}(a, b) &= \sigma_{n_1 \gamma_{h_1}(n_2) h_1 h_2}(a, b) = (\alpha_{n_1 \gamma_{h_1}(n_2)} \delta_{h_1 h_2}(a), \beta_{h_1 h_2}(b)) \\ &= (\alpha_{n_1} \alpha_{\gamma_{h_1}(n_2)} \delta_{h_1} \delta_{h_2}(a), \beta_{h_1} \beta_{h_2}(b)) = (\alpha_{n_1} \delta_{h_1} \alpha_{n_2} \delta_{h_2}(a), \beta_{h_1} \beta_{h_2}(b)) \quad (\text{by (7)}) \\ &= \sigma_{n_1 h_1} \sigma_{n_2 h_2}(a, b). \end{aligned}$$

Thus σ is an action of G on $A \times B$.

Let $\pi : G \rightarrow A \times B$ be given by $\pi(nh) = (\pi_N(n), \pi_H(h))$. Since both $\pi_N : N \rightarrow A$ and $\pi_H : H \rightarrow B$ are bijective, so is π . Moreover

$$\begin{aligned} \pi(n_1 h_1 n_2 h_2) &= \pi(n_1 \gamma_{h_1}(n_2) h_1 h_2) = (\pi_N(n_1 \gamma_{h_1}(n_2)), \pi_H(h_1 h_2)) \\ &= (\pi_N(n_1) \alpha_{n_1} \pi_N \gamma_{h_1}(n_2), \pi_H(h_1) \beta_{h_1}(\pi_H(h_2))) \\ &= (\pi_N(n_1) \alpha_{n_1} \delta_{h_1} \pi_N(n_2), \pi_H(h_1) \beta_{h_1}(\pi_H(h_2))) \\ &= (\pi_N(n_1), \pi_H(h_1)) (\alpha_{n_1} \delta_{h_1} \pi_N(n_2), \beta_{h_1}(\pi_H(h_2))) \\ &= \pi(n_1 h_1) \sigma_{n_1 h_1}(\pi(n_2 h_2)), \end{aligned}$$

for all $n_1, n_2 \in N$ and $h_1, h_2 \in H$. Thus π is a bijective 1-cocycle and we conclude that G is an IYB group. ■

Corollary 3.5 *Let G be an IYB group and H an IYB subgroup of Sym_n . Then the wreath product $G \wr H$ of G and H is an IYB group.*

Proof. Recall that $W = G \wr H$ is the semidirect product $G^n \rtimes H$, where the action of H on G^n is given by $\gamma_h(g_1, \dots, g_n) = (g_{h(1)}, \dots, g_{h(n)})$. Since G is an IYB group, there is an action on an abelian group A which admits a bijective 1-cocycle $\pi : G \rightarrow A$. This action, applied componentwise, induces an action α of G^n on A^n and the map $\bar{\pi} : G^n \rightarrow A^n$, which acts as π component wise, is a bijective 1-cocycle (see the proof of Corollary 3.2). Furthermore the map $\delta : H \rightarrow \text{Aut}(A^n)$, given by $\delta_h(a_1, \dots, a_n) = (a_{h(1)}, \dots, a_{h(n)})$, for all $h \in H$ and $(a_1, \dots, a_n) \in A^n$, is an action of H on A^n such that $\delta_h \bar{\pi} = \bar{\pi} \gamma_h$ for every $h \in H$. Thus W is an IYB group, by Theorem 3.4. ■

Corollary 3.6 *Let n be a positive integer. Then the Sylow subgroups of Sym_n are IYB groups.*

Proof. It is known that the Sylow p -subgroups of Sym_n are isomorphic to a group of the form $G_1 \times G_2$ or $G_3 \wr C_p$, where G_1, G_2, G_3 are Sylow p -subgroup of $\text{Sym}_{m_1}, \text{Sym}_{m_2}, \text{Sym}_{m_3}$ respectively, for some $m_1, m_2, m_3 < n$, [11, pages 10,11]. Since C_p is an IYB group, the result follows from Corollaries 3.2 and 3.5 by induction on n . ■

As a consequence of this result and Corollary 3.2, we have the following result.

Corollary 3.7 *Any finite nilpotent group is isomorphic to a subgroup of an IYB (nilpotent) group.*

■

The next result yields many examples of IYB groups.

Theorem 3.8 *Let G be a finite group having a normal sequence*

$$1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$$

satisfying the following conditions:

- (i) *For every $1 \leq i \leq n$, $G_i = G_{i-1}A_i$ for some abelian subgroup A_i .*
- (ii) *$(G_{i-1} \cap (A_i \cdots A_n), G_{i-1}) = 1$.*
- (iii) *A_i is normalized by A_j for every $i \leq j$.*

Then G is an IYB group.

Proof. Assumption (iii) implies that for every $1 \leq i, j \leq n$, either A_i normalizes A_j or A_j normalizes A_i . Thus $A_i A_j$ is a subgroup of G and $A_i A_j = A_j A_i$. Furthermore $G_i = A_1 \cdots A_i$ and, in particular, $Z = A_1 \cup \cdots \cup A_n$ is a generating subset of G . We set $H_i = A_{i+1} \cdots A_n$ for every $1 \leq i \leq n$. Then A_i normalizes G_j and H_i normalizes A_j for every $j \leq i$. In particular, $G_i \trianglelefteq G$ for every i . Every $g \in G$ can be written, in a non necessarily unique way, as $g = g_{i1}g_{i2}$ with $g_{i1} \in G_i$ and $g_{i2} \in H_i$.

Define $\mu: G \rightarrow \text{Sym}_Z$ by

$$\mu(g)(a) = g_{(i-1)2} a g_{(i-1)2}^{-1},$$

for every $g \in G$ and $a \in A_i$, with $i \geq 1$. We need to show that the map μ is well defined. So let $g = a_1 \cdots a_n = b_1 \cdots b_n$, with $a_j, b_j \in A_j$, for every $j = 1, \dots, n$ and let $a \in A_i$. We can take $g_{i1} = a_1 \cdots a_i \in G_i$ and $g_{i2} = a_{i+1} \cdots a_n \in H_i$, or we can take $\bar{g}_{i1} = b_1 \cdots b_i \in G_i$ and $\bar{g}_{i2} = b_{i+1} \cdots b_n \in H_i$. Since $G_i \trianglelefteq G$ and $g_{i1}g_{i2} = \bar{g}_{i1}\bar{g}_{i2}$,

$$\bar{g}_{i2}^{-1} g_{i2} = \bar{g}_{i2}^{-1} g_{i1}^{-1} \bar{g}_{i1} \bar{g}_{i2} \in G_i \cap H_i.$$

Hence, by condition (ii), $\bar{g}_{i2}^{-1} g_{i2} a g_{i2}^{-1} \bar{g}_{i2} = a$. Thus

$$g_{i2} a g_{i2}^{-1} = \bar{g}_{i2} a \bar{g}_{i2}^{-1} \in A_i.$$

Since A_i is abelian, conjugating by a_i on the left side of the previous equality and by b_i on the right side we have

$$g_{(i-1)2} a g_{(i-1)2}^{-1} = \bar{g}_{(i-1)2} a \bar{g}_{(i-1)2}^{-1}.$$

Suppose that $a \in A_i \cap A_{i'}$ with $i' < i$. Since H_i normalizes both A_i and $A_{i'}$ and A_i is abelian we have

$$g_{(i-1)2} a g_{(i-1)2}^{-1} = a_i g_{i2} a g_{i2}^{-1} a_i^{-1} \in A_{i'} \cap A_i \subseteq G_{i-1} \cap A_i.$$

Since $g_{(i'-1)2} g_{(i-1)2}^{-1} = a_{i'} a_{i'+1} \cdots a_{i-1} \in G_{i-1}$, by (ii),

$$g_{(i-1)2} a g_{(i-1)2}^{-1} = g_{(i'-1)2} a g_{(i'-1)2}^{-1}.$$

Therefore μ is well defined.

Let $g, h \in G$ and $a \in A_i$. Then, with the above notation,

$$\begin{aligned}\mu(gh)(a) &= \mu(g_{i1}g_{i2}h_{i1}h_{i2})(a) = \mu((g_{i1}g_{i2}h_{i1}g_{i2}^{-1})(g_{i2}h_{i2}))(a) \\ &= \mu(g_{i2}h_{i2})(a) = g_{i2}h_{i2}ah_{i2}^{-1}g_{i2}^{-1} = \mu(g)\mu(h)(a).\end{aligned}$$

Hence μ is a group homomorphism.

Let $a \in A_i$ and $b \in A_j$ with $i \leq j$. Then

$$a\mu(a)^{-1}(b) = ab = bb^{-1}ab = b\mu(b)^{-1}(a).$$

By Theorem 2.1, G is an IYB group. ■

Corollary 3.9 *Let G be a finite group. If $G = NA$, where N and A are two abelian subgroups of G and N is normal in G , then G is an IYB group.*

In particular, every abelian-by-cyclic finite group is IYB.

Corollary 3.10 *Every finite nilpotent group of class 2 is IYB.*

Proof. Let G be a finite nilpotent group of class 2. Then there exist $x_1, \dots, x_n \in G$ such that $G/Z(G)$ is the inner direct product of $\langle \bar{x}_1 \rangle, \dots, \langle \bar{x}_n \rangle$, where \bar{x} denotes the class of $x \in G$ modulo its center $Z(G)$. Let $A_i = \langle \{x_i\} \cup Z(G) \rangle$, for all $i = 1, \dots, n$. Let $G_i = A_1 \cdots A_i$, for all $i = 1, \dots, n$. It is easy to check that the group G and the subgroups A_i and G_i satisfy the hypothesis of Theorem 3.8. Hence G is an IYB group. ■

4 Examples

In Section 3 we have given some sufficient conditions for a finite group to be IYB. In this section we present some examples of IYB that are not covered by these results.

Example 4.1 A group of smallest order not satisfying the conditions of Theorem 3.8 is

$$G = Q_8 \rtimes C_3 = \langle x, y, a \mid x^4 = x^2y^2 = a^3 = 1, x^y = x^{-1}, axa^{-1} = y, aya^{-1} = xy \rangle.$$

One may try to show that G is IYB by using Theorem 3.3 and Corollary 3.9. However, the straightforward approach does not work. Nevertheless, we can still show that G is IYB as follows.

Let $Z = \{x, x^{-1}, y, y^{-1}, xy, yx, a\}$ and define $\mu: Z \rightarrow \text{Sym}_Z$ by

$$\begin{aligned}\mu(x) &= \mu(x^{-1}) = (x, x^{-1})(y, y^{-1}), & \mu(y) &= \mu(y^{-1}) = (y, y^{-1})(xy, yx), \\ \mu(xy) &= \mu(yx) = (x, x^{-1})(xy, yx), & \mu(a) &= (x, y, xy)(x^{-1}, y^{-1}, yx).\end{aligned}$$

Note that $\mu(x)\mu(y) = \mu(xy)$, $\mu(x)^4 = \mu(x)^2\mu(y)^2 = \mu(a)^3 = \text{id}$, $\mu(x)^{\mu(y)} = \mu(x^{-1})$, $\mu(a)\mu(x)\mu(a)^{-1} = \mu(y)$, $\mu(a)\mu(y)\mu(a)^{-1} = \mu(x)\mu(y)$. Hence μ extends to a homomorphism $\mu: Q_8 \rtimes C_3 \rightarrow \text{Sym}_Z$. It is easy to check that

$$u\mu(u^{-1})(v) = v\mu(v)^{-1}(u)$$

for all $u, v \in Z$. Therefore, $Q_8 \rtimes C_3$ is an IYB group, by Theorem 2.1. ■

Example 4.2 By Corollary 3.9, every abelian-by-cyclic group is IYB. However it is not clear whether every cyclic-by-abelian group is IYB. In this example we show that some class of cyclic-by-abelian groups consists of IYB groups. It includes all cyclic-by-two generated abelian p -groups.

Consider the following cyclic-by-abelian group of order nq_1q_2 :

$$G = \langle a, b, c \mid a^n = 1, bab^{-1} = a^{r_1}, cac^{-1} = a^{r_2}, b^{q_1} = a^{s_1}, c^{q_2} = a^{s_2}, cbc^{-1} = a^tb \rangle.$$

Then the following conditions hold, where $o_n(r)$ denotes the multiplicative order of r modulo n (for $(r, n) = 1$):

$$\begin{aligned} o_n(r_i) & \mid q_i, & (\text{because } (a, b_i^{q_i}) = 1) \\ s_i(r_i - 1) & \equiv 0 \pmod{n}, & (\text{because } (a^{s_1}, b) = 1 = (a^{s_2}, c)) \\ t(1 + r_1 + r_1^2 + \cdots + r_1^{q_1-1}) & \equiv s_1(r_2 - 1) \pmod{n}, & (\text{because } (a^tb)^{q_1} = ca^{s_1}c^{-1}) \\ -t(1 + r_2 + r_2^2 + \cdots + r_2^{q_2-1}) & \equiv s_2(r_1 - 1) \pmod{n} & (\text{because } (a^{-t}c)^{q_2} = ba^{s_2}b^{-1}). \end{aligned}$$

We also assume that there is an integer u such that

$$r_1 - 1 \equiv u(r_2 - 1) \pmod{n}. \quad (8)$$

Let $A = \langle a \rangle$. Set $Z = \{a, a^2, \dots, a^{n-1}, b, ab, \dots, a^{n-1}b, c\}$, a generating subset of G , and let $f, g \in \text{Sym}_Z$ be given by

$$f : \begin{cases} a^i \mapsto a^{ir_1}, & (1 \leq i < n) \\ a^jb \mapsto a^{tu+r_1j}b, & (0 \leq j < n) \\ c \mapsto c \end{cases}$$

$$g(x) = cxc^{-1}, \text{ for each } x \in Z.$$

We claim that

$$f^{q_1} = g^{q_2} = (f, g) = 1. \quad (9)$$

Since $c^{q_2} = a^{s_2}$ and $s_2(r_2 - 1) \equiv 0 \pmod{n}$ and $s_2(r_1 - 1) \equiv us_2(r_2 - 1) \equiv 0 \pmod{n}$, one has that $c^{q_2} \in Z(G)$ and so $g^{q_2} = 1$. Notice that $f(a^i) = ba^ib^{-1}$ for each i . Since $b^{q_1} \in \langle a \rangle$, one has $f^{q_1}(a^i) = a^i$. On the other hand

$$tu(1 + r_1 + r_1^2 + \cdots + r_1^{q_1-1}) \equiv us_1(r_2 - 1) \equiv s_1(r_1 - 1) \equiv 0 \pmod{n}.$$

Using this and that $o_n(r_1) \mid q_1$, one has

$$f^{q_1}(a^jb) = a^{tu(1+r_1+r_1^2+\cdots+r_1^{q_1-1})+r_1^{q_1}j}b = a^jb.$$

This shows that $f^{q_1} = 1$. Now we check that $gf = fg$. Since the actions of f and g on the powers of a are by conjugation by b and c respectively, the action of (f, g) on these powers is by conjugation by $(b, c) = a^t$ and so $(f, g)(a^i) = a^i$. Finally

$$\begin{aligned} gf(a^jb) &= g(a^{tu+r_1j}b) = a^{(tu+r_1j)r_2+t}b = a^{tu(r_2-1)+tu+r_1r_2j+t}b \\ &= a^{t(r_1-1)+tu+r_1r_2j+t}b = a^{tu+r_1(r_2j+t)}b = f(a^{r_2j+t}b) = fg(a^jb). \end{aligned}$$

This proves the claim.

Since $G/A \cong C_{q_1} \times C_{q_2}$, there is a group homomorphism $\mu : G \rightarrow \text{Sym}_Z$ such that $\mu(a) = \text{id}$, $\mu(b) = f$ and $\mu(c) = g$. Now we check (3) for all $x, y \in Z$. If $x \in \langle a \rangle$, then $\mu(x) = \text{id}$ and $\mu(y)(x) = yxy^{-1}$. Thus $x\mu(x)^{-1}(y) = xy = y(y^{-1}xy) = y\mu(y)^{-1}(x)$ as wanted. If $x = c$ then $\mu(y)(x) = x$ and $\mu(x)(y) = xyx^{-1}$. Again $x\mu(x)^{-1}(y) = xx^{-1}yx = yx = y\mu(y)^{-1}(x)$. By symmetry,

(3) also holds if $y \in \langle a \rangle \cup \{c\}$. If v is an inverse of r_1 modulo n then $f^{-1}(a^j b) = a^{v(j-tu)}b$. Thus, for $x = a^i b, y = a^j b$, one has

$$\begin{aligned} x\mu(x)^{-1}(y) &= xf^{-1}(y) = a^i b a^{v(j-tu)}b = a^{i+r_1 v(j-tu)}b^2 = a^{i+j-tu}b^2 \\ &= a^{j+r_1 v(i-tu)}b^2 = a^j b a^{v(i-tu)}b = yf^{-1}(x) = y\mu(y)^{-1}(x). \end{aligned}$$

We conclude that if condition (8) holds then, by Theorem 2.1, G is an IYB group. Notice that if n is a prime power then, by interchanging the roles of b and c if needed, one may assume that condition (8) holds because the lattice of additive subgroups of \mathbb{Z}_n is linearly ordered. So if G has a normal cyclic p -subgroup A such that G/A is 2-generated and abelian then G is IYB. ■

Example 4.3 By Corollaries 3.9 and 3.10 every nilpotent group of class at most 2 is IYB and every group which is a product of two abelian subgroups, one of them normal, is IYB. In this example we consider 3-group of minimal order not satisfying any of these properties. Namely let G be the group given by the following presentation

$$G = \langle a, b, c, d, e \mid a, b \in Z(G), dc = acd, ec = bce, ed = cde, a^3 = b^3 = c^3 = d^3 = e^3 = 1 \rangle.$$

If $i_1, i_2 \in \{0, 1, 2\}, j_1, j_2 \in \{0, 1\}$, then

$$d^{i_1} e^{j_1} d^{i_2} e^{j_2} = a^{2i_2(i_2-1)j_1+i_1 i_2 j_1} c^{i_2 j_1} d^{i_1+i_2} e^{j_1+j_2}. \quad (10)$$

Let $H = \langle G', d \rangle$ and $Z = H \cup He$. Let $D, E \in S_Z$ be given by

$$D(nd^i e^j) = dnd^{-1}a^{i+2ij}b^{2i}c^{2j}d^i e^j \quad \text{and} \quad E(nd^i e^j) = ene^{-1}d^{i+j}e^j, \quad (n \in G', i = 0, 1, 2; j = 0, 1).$$

If $n \in G', i = 0, 1, 2$ and $j = 0, 1$ then

$$D^k(nd^i e^j) = D^k(n)D^k(d^i)D^k(e^j) = d^k n d^{-k} a^{k(i+2ij)+k(k-1)j} b^{2ik} c^{2kj} d^i e^j \quad (11)$$

and

$$E^k(nd^i e^j) = E^k(n)E^k(d^i)E^k(e^j) = e^k n e^{-k} d^{i+kj} e^j, \quad (12)$$

for all positive integer k . Moreover,

$$\begin{aligned} ED(nd^i e^j) &= E(dnd^{-1}a^{i+2ij}b^{2i}c^{2j}d^i e^j) = ednd^{-1}e^{-1}a^{i+2ij}b^{2(i+j)}c^{2j}d^{i+j}e^j \\ &= c(dene^{-1}d^{-1})c^{-1}a^{i+2ij}b^{2(i+j)}c^{2j}d^{i+j}e^j = dene^{-1}d^{-1}a^{i+2ij}b^{2(i+j)}c^{2j}d^{i+j}e^j \\ &= D(ene^{-1}d^{i+j}e^j) = DE(nd^i e^j) \end{aligned}$$

Thus $ED = DE$ and $D^3 = E^3 = 1$ and therefore there is a unique group homomorphism $\mu : G \rightarrow \text{Sym}_Z$ such that $\mu(G') = 1, \mu(d) = D$ and $\mu(e) = E$. We will check (3) for all $x, y \in Z$. Let

$x = m_1 d^{i_1} e^{j_1}$ and $y = m_2 d^{i_2} e^{j_2}$, with $m_1, m_2 \in G'$, $i_1, i_2 \in \{0, 1, 2\}$ and $j_1, j_2 \in \{0, 1\}$. We have

$$\begin{aligned}
x\mu(x)^{-1}(y) &= m_1 d^{i_1} e^{j_1} \mu(m_1 d^{i_1} e^{j_1})^{-1} (m_2 d^{i_2} e^{j_2}) \\
&= m_1 m_2 d^{i_1} e^{j_1} E^{2j_1} D^{2i_1} (d^{i_2} e^{j_2}) && \text{(by (11) and (12))} \\
&= m_1 m_2 d^{i_1} e^{j_1} E^{2j_1} (a^{2i_1(i_2+2i_2j_2)+2i_1(2i_1-1)j_2} b^{i_1i_2} c^{i_1j_2} d^{i_2} e^{j_2}) && \text{(by (11))} \\
&= m_1 m_2 d^{i_1} e^{j_1} a^{2i_1(i_2+2i_2j_2)+2i_1(2i_1-1)j_2} b^{i_1i_2+2i_1j_1j_2} c^{i_1j_2} d^{i_2+2j_1j_2} e^{j_2} && \text{(by (12))} \\
&= m_1 m_2 a^{2i_1(i_2+2i_2j_2)+2i_1(2i_1-1)j_2+i_1^2j_2} b^{i_1i_2} c^{i_1j_2} d^{i_1} e^{j_1} d^{i_2+2j_1j_2} e^{j_2} \\
&= m_1 m_2 a^{2i_1(i_2+2i_2j_2)+2i_1(2i_1-1)j_2+i_1^2j_2+2(i_2+2j_1j_2)(i_2+2j_1j_2-1)j_1+i_1(i_2+2j_1j_2)j_1} \\
&\quad \cdot b^{i_1i_2} c^{i_1j_2+(i_2+2j_1j_2)j_1} d^{i_1+i_2+2j_1j_2} e^{j_1+j_2} && \text{(by (10))} \\
&= m_1 m_2 a^{2i_1i_2+i_1i_2j_2+2i_1^2j_2+i_1j_2+2i_2^2j_1+2i_2j_1^2j_2+2j_1^3j_2^2-2i_2j_1-j_1^2j_2+i_1i_2j_1+2i_1j_1^2j_2} \\
&\quad \cdot b^{i_1i_2} c^{i_1j_2+i_2j_1+2j_1^2j_2} d^{i_1+i_2+2j_1j_2} e^{j_1+j_2} \\
&= m_1 m_2 a^{2i_1i_2+i_1i_2j_2+2i_1^2j_2+i_1j_2+2i_2^2j_1+2i_2j_1j_2+2j_1j_2-2i_2j_1-j_1j_2+i_1i_2j_1+2i_1j_1j_2} \\
&\quad \cdot b^{i_1i_2} c^{i_1j_2+i_2j_1+2j_1j_2} d^{i_1+i_2+2j_1j_2} e^{j_1+j_2} && \text{(since } j_1^2 = j_1 \text{ and } j_2^2 = j_2) \\
&= m_1 m_2 a^{2i_1i_2+i_1i_2(j_1+j_2)+2(i_1^2j_2+i_2^2j_1)+i_1j_2+i_2j_1+2(i_1+i_2)j_1j_2+j_1j_2} \\
&\quad \cdot b^{i_1i_2} c^{i_1j_2+i_2j_1+2j_1j_2} d^{i_1+i_2+2j_1j_2} e^{j_1+j_2}
\end{aligned}$$

This expression is invariant by interchanging m_1 and m_2 , i_1 and i_2 , and j_1 and j_2 . Hence it follows that $x\mu(x)^{-1}(y) = y\mu(y)^{-1}(x)$. By Theorem 2.1, G is IYB. ■

Let μ be an IYB morphism of G . Then $\ker(\mu)$ is abelian. If $x \in G$ and $k \in \ker(\mu)$ then

$$\mu(x)(k) = xx^{-1}\mu(x^{-1})^{-1}(k) = xk\mu(k)(x^{-1}) = xkx^{-1}. \quad (13)$$

This implies, by (5), that if N is a normal subgroup of G contained in $\ker(\mu)$ then $\mu(x)(Ny) = N\mu(x)(y)$ and therefore (with the usual bar notation) the map $\bar{\mu} : \bar{G} = G/N \rightarrow S_{\bar{G}}$ given by

$$\bar{\mu}(\bar{x})(\bar{y}) = \overline{\mu(x)(y)}$$

is well defined and it is easy to show that $\bar{\mu}$ is an IYB morphism of \bar{G} . We say that μ is a lifting of $\bar{\mu}$.

It is somehow natural to try to prove that every solvable group is IYB with the following induction strategy: Let G be a non-trivial solvable group. Take a non-trivial abelian normal subgroup N of G . Assume, by induction, that $\bar{G} = G/N$ has an IYB morphism λ and prove that λ admits a lifting to G , i.e. $\lambda = \bar{\mu}$ for some IYB morphism μ of G . Notice that if this strategy works then every non-trivial solvable group should have a non injective IYB morphism. This is the case for all the examples of IYB groups G that we have computed. This leads us to the following natural question: Does every IYB group admit a non-injective IYB morphism? In fact, all the IYB morphisms which appears implicitly in the results of Section 3 or in the above examples are non-injective and therefore they are liftings of IYB morphisms of proper quotients. These examples may lead to the impression that every IYB morphism of G/N , for an abelian normal subgroup N of G , can be lifted to an IYB morphism of G . However this is false as the following example shows.

Example 4.4 Let G be the group of Example 4.3. We claim that the trivial IYB morphism of G/G' does not lift to an IYB morphism of G . Indeed, assume that μ is an IYB morphism of G

which lifts the trivial IYB morphism of G/G' . Then, clearly, $\mu(d)$ and $\mu(e)$ commute. Furthermore, by (13), $\mu(g)(n) = gng^{-1}$ for every $g \in G$ and $n \in G'$ and there are $x_j \in G'$, ($j = 1, 2, 3$), with

$$\mu(d)(d) = x_1d, \quad \mu(e)(d) = x_2d, \quad \mu(e)(e) = x_3e. \quad (14)$$

Then, by (5),

$$dx_2d^{-1}x_1d = \mu(d)(x_2d) = \mu(d)\mu(e)(d) = \mu(e)\mu(d)(d) = \mu(e)(x_1d) = ex_1e^{-1}x_2d.$$

Therefore $(e, x_1) = (d, x_2) \in \langle a \rangle \cap \langle b \rangle = \{1\}$ and hence $x_1, x_2 \in Z(G)$. This implies, by (5) and (13), that $\mu(d^j)(d^k) = x_1^{jk}d^k$ and $\mu(e^j)(d^k) = x_2^{jk}d^k$, for every j, k . Then $\mu(d)(e) = dd^{-1}\mu(d^{-1})^{-1}(e) = de\mu(e)^{-1}(d^{-1}) = dex_2d^2 = x_2c^2e$. Thus

$$\mu(e)\mu(d)(e) = \mu(e)(x_2c^2e) = x_2b^2c^2x_3e$$

and

$$\mu(d)\mu(e)(e) = \mu(d)(x_3e) = dx_3d^{-1}x_2c^2e.$$

Therefore $b^2x_3 = dx_3d^{-1}$ and so $b^2 = (d, x_3) \in \langle a \rangle$, a contradiction. ■

5 Solutions to the Yang-Baxter equation associated to one IYB group

Let X be a finite set. We have seen in Section 2 that if $\lambda: X \rightarrow \text{Sym}_X$ is an IYB map, then the map

$$r: X \times X \longrightarrow X \times X$$

defined by $r(x, y) = (\lambda(x)(y), \lambda(\lambda(x)(y))^{-1}(x))$, for all $x, y \in X$, is an involutive non degenerate set theoretical solution of the Yang-Baxter equation. We also know, by Theorem 2.1, that $\langle \lambda(X) \rangle$ is an IYB group. One can ask how many involutive non degenerate set theoretical solutions of the Yang-Baxter equation are associated to the same IYB group. Note that for any finite set X , the map $\lambda: X \rightarrow \text{Sym}_X$ defined by $\lambda(x) = \text{id}_X$ for all $x \in X$ is an IYB map associated to the trivial group. Note also that if $\lambda_i: X_i \rightarrow \text{Sym}_{X_i}$ is an IYB map associated to the IYB group G_i , for $i = 1, \dots, n$, then it is easy to see that if $X = \bigcup_{i=1}^n X_i$ is a disjoint union, then the map $\lambda: X \rightarrow \text{Sym}_X$ defined by $\lambda(x) = \lambda_i(x)$, for all $x \in X_i$, is an IYB map associated to direct product $\prod_{i=1}^n G_i$, considering Sym_{X_i} naturally included in Sym_X . Notice that this also gives another proof for Corollary 3.2. Hence, since $\langle 1 \rangle \times G \cong G$, one can associate with each IYB group, in an obvious way, infinitely many involutive non degenerate set theoretical solutions of the Yang-Baxter equation. Now we give a non-obvious construction of an infinite family of involutive non degenerate set theoretical solutions of the Yang-Baxter equation associated to a fixed IYB group.

Lemma 5.1 *Let X be a set and let $\lambda: X \rightarrow \text{Sym}_X$ be a map. Let $\psi: \text{Sym}_X \rightarrow \text{Sym}_{X^2}$ be the map defined by*

$$\psi(\tau)(x, y) = (\tau(x), \lambda(\tau(x))^{-1}\tau\lambda(x)(y))$$

for all $\tau \in \text{Sym}_X$ and $x, y \in X$. Then ψ is a monomorphism.

Proof. Let $\tau_1, \tau_2 \in \text{Sym}_X$ and $x, y \in X$. We have:

$$\begin{aligned} \psi(\tau_1\tau_2)(x, y) &= (\tau_1\tau_2(x), \lambda(\tau_1\tau_2(x))^{-1}\tau_1\tau_2\lambda(x)(y)) \\ &= (\tau_1(\tau_2(x)), \lambda(\tau_1(\tau_2(x)))^{-1}\tau_1\lambda(\tau_2(x))(\lambda(\tau_2(x))^{-1}\tau_2\lambda(x)(y))) \\ &= \psi(\tau_1)(\tau_2(x), \lambda(\tau_2(x))^{-1}\tau_2\lambda(x)(y)) \\ &= \psi(\tau_1)\psi(\tau_2)(x, y). \end{aligned}$$

Hence ψ is a homomorphism. It is easy to see that it is injective. ■

Lemma 5.2 *Let $\lambda: X \rightarrow \text{Sym}_X$ be an IYB map. Let $\mu: X^2 \rightarrow \text{Sym}_X$ be the map defined by $\mu(x, y) = \lambda(x)\lambda(y)$ for all $x, y \in X$. Let $\lambda_2: X^2 \rightarrow \text{Sym}_{X^2}$ be the map defined by $\lambda_2 = \psi\mu$, where ψ is as in Lemma 5.1. Then λ_2 is an IYB map.*

Proof. Note that

$$\lambda_2(x, y)^{-1}(z, t) = (\lambda(y)^{-1}\lambda(x)^{-1}(z), \lambda(\lambda(y)^{-1}\lambda(x)^{-1}(z))^{-1}\lambda(y)^{-1}\lambda(x)^{-1}\lambda(z)(t)),$$

for all $x, y, z, t \in X$.

We have

$$\begin{aligned} \lambda_2(x, y)\lambda_2(\lambda_2(x, y))^{-1}(z, t) \\ = \psi(\lambda(x)\lambda(y)\lambda(\lambda(y)^{-1}\lambda(x)^{-1}(z))\lambda(\lambda(\lambda(y)^{-1}\lambda(x)^{-1}(z))^{-1}\lambda(y)^{-1}\lambda(x)^{-1}\lambda(z)(t))). \end{aligned}$$

So, in order to prove the lemma, we need to show that

$$\begin{aligned} \lambda(x)\lambda(y)\lambda(\lambda(y)^{-1}\lambda(x)^{-1}(z))\lambda(\lambda(\lambda(y)^{-1}\lambda(x)^{-1}(z))^{-1}\lambda(y)^{-1}\lambda(x)^{-1}\lambda(z)(t)) \\ = \lambda(z)\lambda(t)\lambda(\lambda(t)^{-1}\lambda(z)^{-1}(x))\lambda(\lambda(\lambda(t)^{-1}\lambda(z)^{-1}(x))^{-1}\lambda(t)^{-1}\lambda(z)^{-1}\lambda(x)(y)). \end{aligned} \quad (15)$$

Since λ is an IYB map,

$$\begin{aligned} \lambda(x)\lambda(y)\lambda(\lambda(y)^{-1}\lambda(x)^{-1}(z)) &= \lambda(x)\lambda(\lambda(x)^{-1}(z))\lambda(\lambda(\lambda(x)^{-1}(z))^{-1}(y)) \\ &= \lambda(z)\lambda(\lambda(z)^{-1}(x))\lambda(\lambda(\lambda(x)^{-1}(z))^{-1}(y)). \end{aligned}$$

Hence (15) is equivalent to

$$\begin{aligned} \lambda(\lambda(\lambda(x)^{-1}(z))^{-1}(y))\lambda(\lambda(\lambda(y)^{-1}\lambda(x)^{-1}(z))^{-1}\lambda(y)^{-1}\lambda(x)^{-1}\lambda(z)(t)) \\ = \lambda(\lambda(\lambda(z)^{-1}(x))^{-1}(t))\lambda(\lambda(\lambda(t)^{-1}\lambda(z)^{-1}(x))^{-1}\lambda(t)^{-1}\lambda(z)^{-1}\lambda(x)(y)). \end{aligned} \quad (16)$$

Since λ is an IYB map, to prove (16), it is sufficient to see that

$$\lambda(\lambda(\lambda(x)^{-1}(z))^{-1}(y))(\lambda(\lambda(y)^{-1}\lambda(x)^{-1}(z))^{-1}\lambda(y)^{-1}\lambda(x)^{-1}\lambda(z)(t)) = \lambda(\lambda(z)^{-1}(x))^{-1}(t), \quad (17)$$

Since λ is an IYB map,

$$\begin{aligned} \lambda(\lambda(\lambda(x)^{-1}(z))^{-1}(y))\lambda(\lambda(y)^{-1}\lambda(x)^{-1}(z))^{-1}\lambda(y)^{-1}\lambda(x)^{-1}\lambda(z)(t) \\ = \lambda(\lambda(x)^{-1}(z))^{-1}\lambda(x)^{-1}\lambda(z)(t) \\ = \lambda(\lambda(z)^{-1}(x))^{-1}(t). \end{aligned}$$

Hence (17) is true and the lemma is proved. ■

Note that, with the notation of Lemma 5.2, if $1 \in \lambda(X)$ then $\langle \lambda(X) \rangle \cong \langle \lambda_2(X^2) \rangle$. Thus in this way we can construct infinitely many involutive non degenerate set theoretical solutions of the Yang-Baxter equation associated to the IYB group $\langle \lambda(X) \rangle$.

It seems a difficult problem to describe all the involutive non degenerate set theoretical solutions of the Yang-Baxter equation associated to a given IYB group.

References

- [1] N. Ben David and Y. Ginosar, On groups of central type, non-degenerate and bijective cohomology classes, Israel J. Math. to appear, ArXiv: 0704.2516v1 [math.GR].
- [2] F. Cedó, E. Jespers and J. Okniński, The Gelfand-Kirillov dimension of quadratic algebras satisfying the cyclic condition, Proc. AMS 134 (2005), 653-663.
- [3] V. G. Drinfeld, On unsolved problems in quantum group theory. Quantum Groups, Lecture Notes Math. 1510, Springer-Verlag, Berlin, 1992, 1-8.
- [4] P. Etingof and S. Gelaki, A method of construction of finite-dimensional triangular semi-simple Hopf algebras, Mathematical Research Letters 5 (1998), 551-561.
- [5] P. Etingof, T. Schedler and A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation, Duke Math. J. 100 (1999), 169-209.
- [6] T. Gateva-Ivanova, A combinatorial approach to the set-theoretic solutions of the Yang-Baxter equation, J. Math. Phys. 45 (2004), 3828-3858.
- [7] T. Gateva-Ivanova and M. Van den Bergh, Semigroups of I -type, J. Algebra 206 (1998), 97-112.
- [8] E. Jespers and J. Okniński, Monoids and Group of I -Type, Algebr. Represent. Theory 8 (2005), 709-729.
- [9] E. Jespers and J. Okniński, *Noetherian Semigroup Rings*, Springer, Dordrecht 2007.
- [10] C. Kassel, *Quantum Groups*, Graduate Text in Mathematics 155, Springer-Verlag, New York, 1995.
- [11] D. S. Passman, *Permutation Groups*, Benjamin, New York, 1968.
- [12] D. K. Robinson, *A course in the theory of groups*, second edition, Springer-Verlag, New York, 1996.
- [13] W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, Adv. Math. 193 (2005), 40-55.
- [14] C.N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, Phys. Rev. Lett. 19 (1967), 1312-1315.

<p>F. Cedó Departament de Matemàtiques Universitat Autònoma de Barcelona 08193 Bellaterra (Barcelona), Spain</p>	<p>E. Jespers Department of Mathematics Vrije Universiteit Brussel Pleinlaan 2, 1050 Brussel, Belgium</p>	<p>Á. del Río Departamento de Matemáticas Universidad de Murcia 30100 Murcia, Spain</p>
---	--	--